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THE M-TRAVELLING SALESMEN PROBLEM.(U)
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Research Report CCS 381

THE M-TRAVELLING SALESMEN PROBLEM

by

I. Ali
J. Kennington*

CENTER FOR CYBERNETIC STUDIES

The University of Texas
Austin, Texas 78712

Research Report CCS 381

THE M-TRAVELLING SALESMEN PROBLEM

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I. Ali
J. Kennington*

July 1980

*Southern Methodist University

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CENTER FOR CYBERNETIC STUDIES

A. Charnes, Director
Business-Economics Building, 203E
The University of Texas at Austin
Austin, Texas 78712
(512) 471-1821

ABSTRACT

This paper develops the number of feasible solutions (m-tours) for both the asymmetric and symmetric m-travelling salesmen problems. The proofs are constructive and it is shown how these results may be used to design an enumeration tree for incorporation within a general branch-and-bound algorithm. In addition, for the m-travelling salesmen problem defined over the Euclidean vector space in which the number of salesmen used is a decision variable, it is shown that there exists an optimal tour using only one salesman.

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I. INTRODUCTION

One of the most extensively studied models in the operations research literature is the classical travelling salesman problem (see [2, 3, 4, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 22, 24, 25, 27, 28, 31, 32, 34, 35, 36]). Two generalizations of this model which allow for multiple salesmen have also been developed (see [1, 5, 12, 20, 24, 26, 29, 30, 33]). Let m denote the number of salesmen available for assignment and define a *salesman's tour* (also called a *tour* in this paper) as a sequence of distinct cities $\{n, i_1, i_2, \ldots, i_\ell\}$ where $\ell \geq 1$. These generalizations differ as to whether exactly m or at most m salesmen must be assigned a tour. For the case in which at most m salesmen must be used, the number of salesmen becomes a decision variable.

We now formally state the two models addressed in this exposition. Given n cities (1, ..., n) and m salesmen, all located at city n, we wish to find a set of tours such that each city other than n is a member of exactly one tour. Let c_{ij} denote the distance from city i to city j and let $c_{ni_1} + c_{i_1 i_2} + \ldots + c_{i_{\ell-1} i_\ell} + c_{i_\ell n}$ denote the distance for the salesman's tour $\{n, i_1, i_2, \ldots, i_\ell\}$. The objective is to make salesman assignments so that the total distance travelled by all salesmen is a minimum. If the optimal solution must be composed of m tours, then we call this model the fixed m-travelling salesmen problem (FMTSP). If m is an upper bound on the number of tours in the optimal solution, then we call this model the variable m-travelling salesmen problem (VMTSP). For the variable m case there may be fixed charges associated with the various salesmen. Clearly, for m = 1, both of these models specialize to the classical travelling salesman model. If the distances $c_{ij} = c_{ji}$ for all city pairs, then these models are called *symmetric*; otherwise, they are called *asymmetric*.

The two generalized models discussed above were first formulated in integer programming terms by Miller, Tucker, and Zemlin [24]. Later, Bellmore and Hong [5] proved that the asymmetric versions of FMTSP and VMTSP are each equivalent to asymmetric travelling salesman problems (TSP) defined on n+m-1 cities. Extensions of these results for the symmetric VMTSF have been presented by Hong and Padberg [20] and by Rao [29].

This exposition presents new results which may be used to aid in the development of algorithms which attack these generalized models directly. The major results are as follows:

(i) The number of feasible solutions for a n city asymmetric fixed m-travelling salesmen problem having m salesmen is

$$\binom{n-1}{m} \frac{(n-2)!}{(m-1)!} .$$

(ii) The number of feasible solutions for a n city symmetric fixed m-travelling salesmen problem having m salesmen is

$$\sum_{i=m_1}^{m_2} \left\{ \binom{n-1}{m+i} \binom{m+i}{m-i} \prod_{j=1}^{i} \left[(2i - 2j + 1) \right] \frac{(n-m-2)!}{(i-1)!} \right\}$$

where $m_1 = \min(1, n-m-1)$, $m_2 = \min(m, n-m-1)$, $\prod_{j=1}^{0}(\omega) = 1$ for any ω, and $-1! = 0! = 1$.

(iii) If the triangle inequality is satisfied, then there exists an optimal solution to the variable m-travelling salesmen problem having one salesman's tour.

The first two results may be used to design an enumeration tree for incorporation within a branch-and-bound algorithm (e.g. [1, 12, 33]). The third result indicates that any VMTSP defined on a Euclidean vector space *degenerates to the single salesman case.*

II. FIXED M-TRAVELLING SALESMAN PROBLEM

2.1 Asymmetric Case

It is well known that there are $(n-1)!$ tours for an n city asymmetric TSP on a complete graph. Therefore, using the results of Bellmore and Hong [5], the one salesman equivalent of the asymmetric FMTSP has $(n+m-2)!$ possible tours on a complete graph. Some of these tours are not feasible for FMTSP and are eliminated by using infinite cost on some of the arcs. The following theorem presents the exact number of feasible solutions for the asymmetric fixed m-travelling salesman problem.

Theorem 1.

Let $n > m$. The number of feasible solutions for the n city asymmetric FMTSP having m salesmen is

$$\binom{n-1}{m} \frac{(n-2)!}{(m-1)!} .$$

Proof: The proof is constructive. Any m-tour may be constructed in the following manner: Select m nodes to be the initial nodes visited by the m salesmen. These may be selected in $\binom{n-1}{m}$ ways. Each of these selections may be displayed as shown in Figure 1a. This structure may be viewed as consisting of m chains $(C_1, \ldots, C_m)$ beginning at city n and n-m-1 other chains $(F_1, \ldots, F_{n-m-1})$, which do not include city n as illustrated in Figure 1b.

Figure 1 About Here

Suppose chain F_{n-m-1} is appended to the end of one of the other chains. This chain may be appended to a chain fixed to n in m ways and may be appended to the end of a chain not fixed to n in n-m-2 ways for a total of $m+n-m-2 = n-2$ ways. After renumbering, the chains may be displayed as shown in either Figure 2a or 2b. For either case there will be m chains $C_1, \ldots, C_m$ and $n-m-2$ chains $F_1, \ldots, F_{n-m-2}$. Suppose chain F_{n-m-2} is appended to the end of one of the other chains. This may be accomplished in $n-3$ ways to produce a new set of chains $C_1, \ldots, C_m$ and $F_1, \ldots, F_{n-m-3}$. This procedure can occur exactly $n-m-1$ times yielding the m chains fixed to n. A feasible m-tour is obtained by connecting the last city in each of the m chains to the base city n. This gives $(n-2)(n-3) \ldots (m)$ possible tours for each of the $\binom{n-1}{m}$ initial selections.

Figure 2 About Here

Hence, the total number of m-tours on n cities is given by

$$\binom{n-1}{m} (n-2)(n-3) \ldots m = \binom{n-1}{m} \frac{(n-2)!}{(m-1)!} . \tag{1}$$

This completes the proof of Theorem 1.

A 5 city 2 salesman enumeration tree using the above construction is illustrated in Figure 3.

Figure 3 About Here

A comparison of the number of feasible m-tours versus the number of 1-tours for the equivalent one salesman problem is given in Table 1. For 9 cities and 3 salesmen, there are 141,120 possible 3-tours versus 3,628,800 possible 1-tours for the equivalent one salesman problem on a complete graph. Based on this analysis, it may be advantageous to attack FMTSP directly rather than its one salesman equivalent.

Table 1 About Here

Note that for values of $n \geq 5$, as m increases, the number of feasible solutions increase to a maximum and then decreases to 1. This is counter-intuitive to the computational experience ([1, 33]) that holding all other parameters fixed, the problems become easier.

2.2 Symmetric Case

Also of interest is the number of m-tours for the symmetric FMTSP. The following theorem gives this result.

Theorem 2

Let $n > m$. The total number of distinct m-tours for the n city symmetric FMTSP having m salesmen is

$$\sum_{i=m_1}^{m_2} \left\{ \binom{n-1}{m+i}\binom{m+i}{m-i} \left[\prod_{j=1}^{i} (2i-2j+1) \right] \frac{(n-m-2)!}{(i-1)!} \right\},$$

where $m_1 = \min(1, n-m-1)$, $m_2 = \min(m, n-m-1)$, $\prod_{j=1}^{0} (\cdot) = 1$, and $-1! = 0! = 1$.

Proof: We say that a tour having city n and one additional city is a singleton tour while a tour having city n and two additional cities is a doubleton tour. The proof is constructive with the initial city assignment involving only singleton and doubleton tours.

Depending on the values of n and m, the number of non-singleton tours varies as follows:

If $0 \leq n-m-1 < m$, then the number of non-singleton tours varies from $\min(1, n-m-1)$ to $n-m-1$.

If $n-m-1 \geq m$, then the number of non-singleton tours varies from 1 to m.

Therefore, for any selection of n and m the total number of non-singleton tours is bounded below by $m_1 = \min(1, n-m-1)$ and above by $m_2 = \min(m, n-m-1)$.

Let $i \varepsilon \{m_1, m_1+1, \ldots, m_2\}$. We now determine the number of ways in which m tours can be constructed such that i tours are doubletons and m-i tours are singletons. For these tours we select $m-i + 2i = m+i$ cities from the n-1 available cities. This can be done in $\binom{n-1}{m+i}$ ways. Given m+i cities, the m-i singletons can be selected in $\binom{m+i}{m-i}$ ways. This leaves i doubletons to be selected from 2i cities. Since the distances are symmetric, this can be accomplished in $\prod_{j=1}^{i} (2i - 2j + 1)$ ways. This gives

$$\binom{n-1}{m+i} \binom{m+i}{m-i} \prod_{j=1}^{i} (2i - 2j + 1) \tag{2}$$

ways to construct m tours such that i are doubletons and m-i are singletons. If $i = 0$, this expression reduces to $\binom{n-1}{m}$.

This leaves n-m-i-1 cities to be appended to the i doubletons. Using the same logic used to develop (1), this may be done in

$$\frac{(n-m-2)!}{(i-1)!} \tag{3}$$

ways where $-1! = 0! = 1$. Taking the product of (2) and (3) and summing over all selections of i yields the desired result. This completes the proof of Theorem 2.

The total number of feasible solutions for various symmetric FMTSP is given in Table 2.

Table 2 About Here

Note that for an n=9, m=3 problem, there are $\frac{141,120}{35,280} = 4$ times as many tours for the asymmetric case as for the symmetric case. Different values of m an n yield different factors. For the m=1 case, the factor is, of course, always 2.

III. VARIABLE M-TRAVELLING SALESMAN PROBLEM

Let i, j, and k denote any cities with travel distances of c_{ij}, c_{ik}, and c_{kj}. We say that a TSP satisfies the *triangle inequality* if $c_{ij} \leq c_{ik} + c_{kj}$ for all cities i, j, and k. For a TSP defined on the Euclidean vector space, the triangle inequality is automatically satisfied. We now show that there exists an optimal solution to the variable m-travelling salesmen problem with one salesman's tour.

Theorem 3

If the triangle inequality is satisfied for an n city VMTSP having m salesman, then there exists an optimal solution having only one salesman's tour.

Proof: Suppose the optimal solution for VMTSP uses $m^* > 1$ salesmen. Let $R_1 = \{(n, i_1), (i_1, i_2), \ldots, (i_\ell, n)\}$ and $R_2 = \{(n, j_1), (j_1, j_2), \ldots, (j_p, n)\}$ denote two of these tours. Then $R_3 = R_1 \cup R_2 \cup \{(i_\ell j_1)\} - \{(i_\ell, n), (n_1, j_1)\}$ is a tour. Since $c_{i_\ell j_1} \leq c_{i_\ell n} + c_{n j_1}$, the length of R_3 is no greater than the length of the sum of the lengths of R_1 and R_2. Therefore, there is an alternate optima using m^*-1 tours. Repeating this argument a total of m^*-1 times yields an optimal solution with one tour. This completes the proof of Theorem 3.

Note that this result holds for both the asymmetric and symmetric problem as well as the problem with nonnegative fixed charges associated with the m salesmen.

IV. SUMMARY AND CONCLUSIONS

This paper presents several new results which can be used to directly enhance existing branch-and-bound algorithms for the m-travelling salesmen problem (such as Svetska and Huckfeldt [33] and Gavish and Srikanth [12]). In addition, it is shown that for larger values of n, the number of feasible solutions increases as m increases and then decreases to one. Unlike the single salesman case, for given values of n and m the total number of tours for the asymmetric problem can be substantially larger (4 or more) than the number of tours for the symmetric case. We believe that this calls for specialized algorithms for these two cases. Finally, we show that the model in which the number of salesmen is a variable degenerates to the one salesman case whenever the triangle inequality holds.

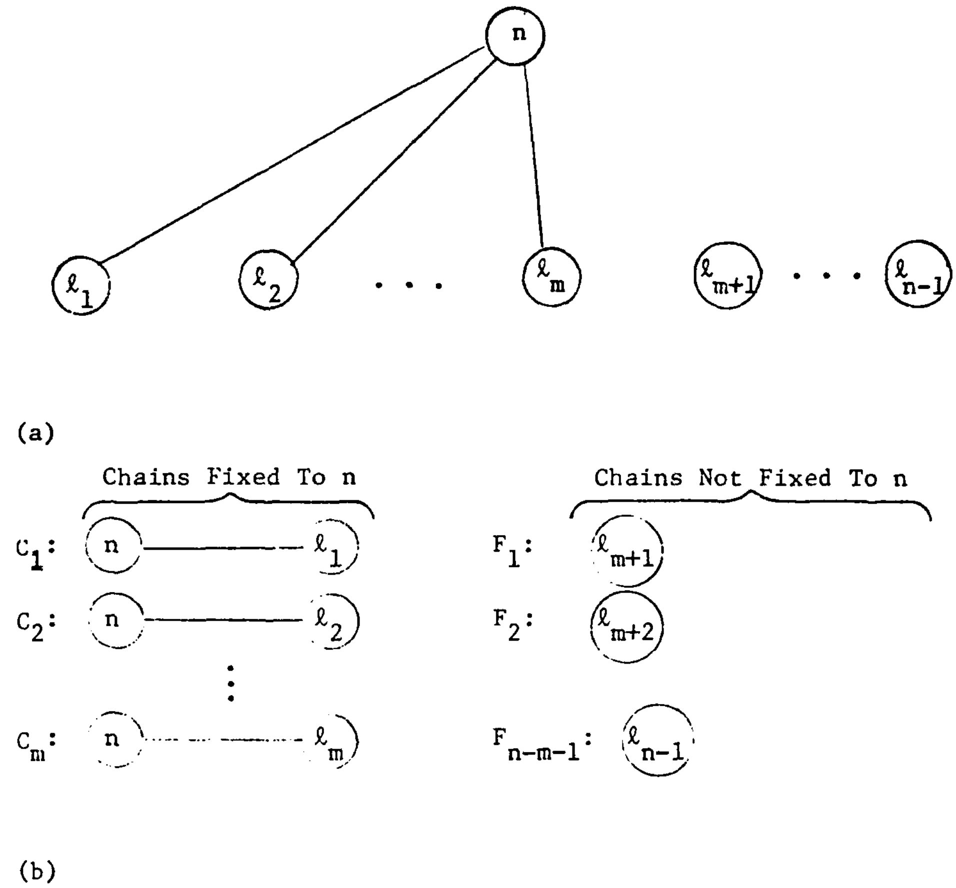

Figure 1. Construction After Selecting The First City In Each Salesman's Tour

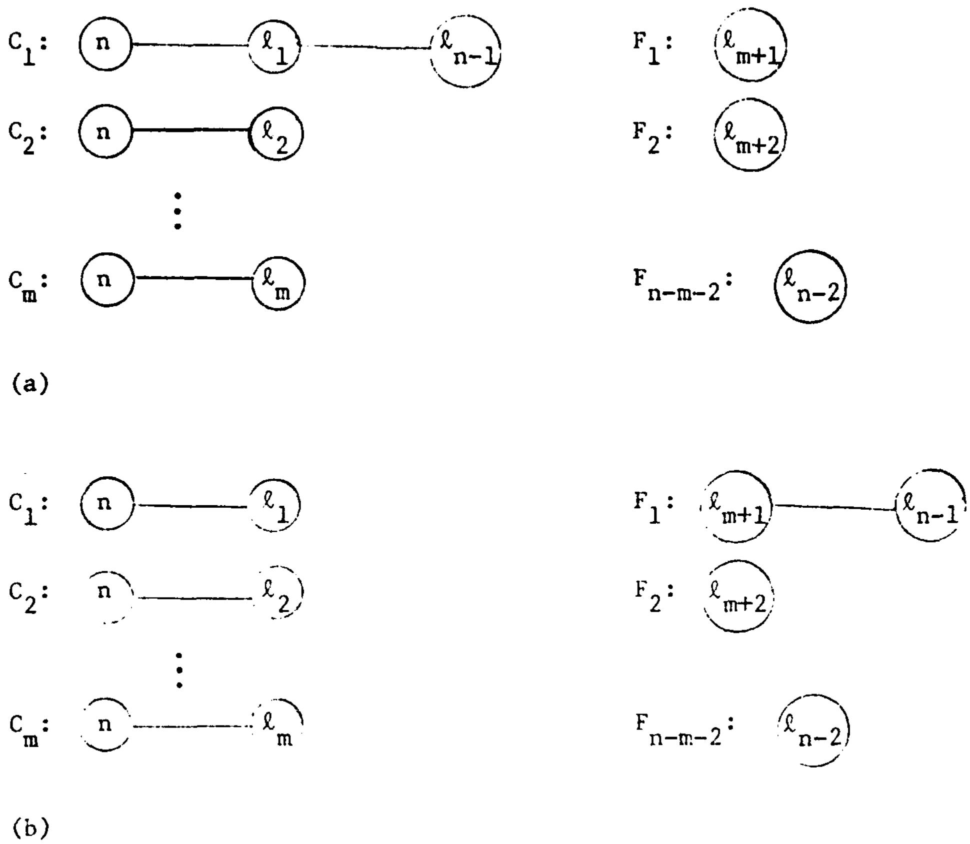

Figure 2. Construction After Appending The Last Chain To The End Of Another Chain

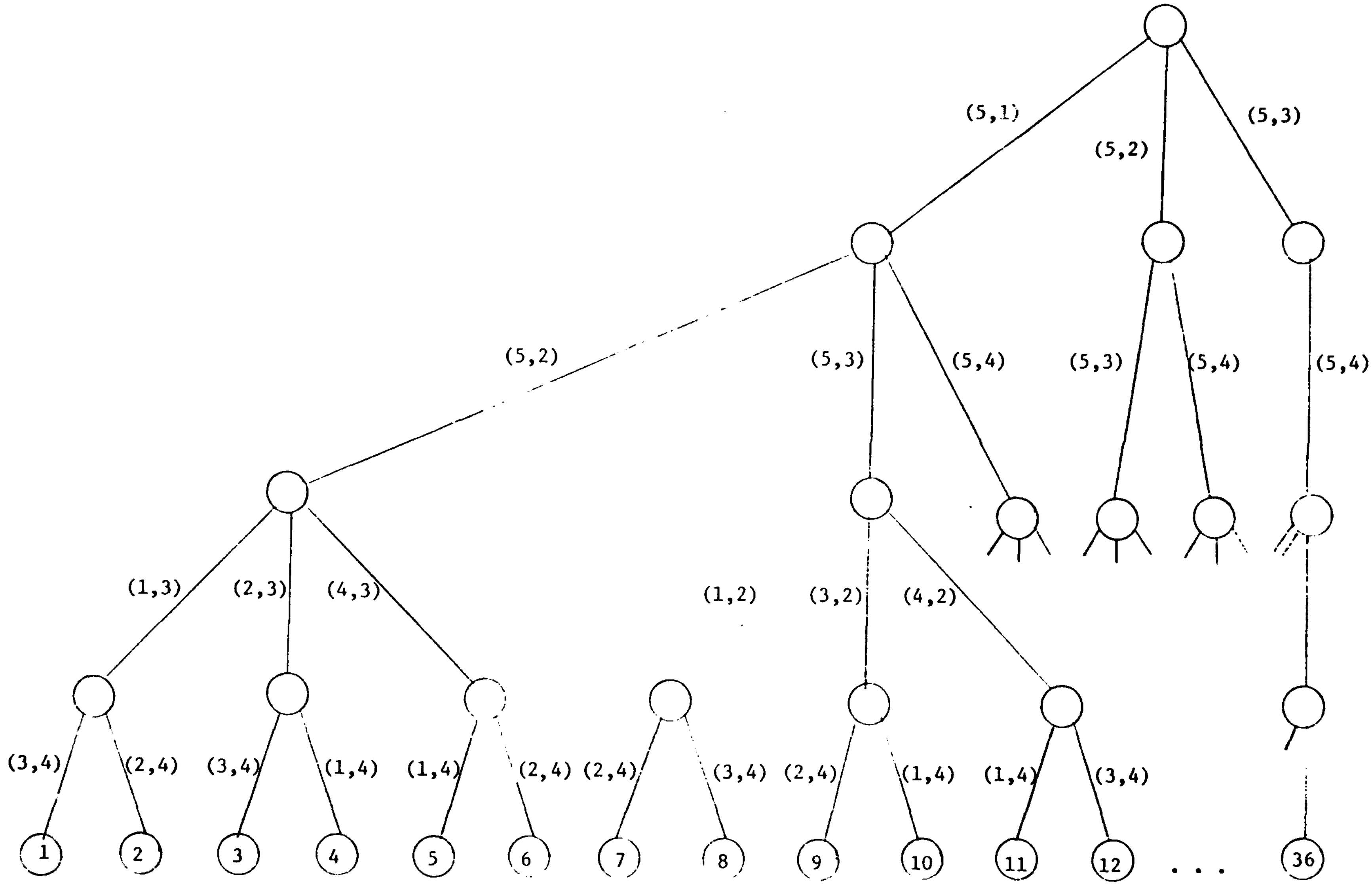

Figure 3. Enumeration Tree For n = 5, m = 2.

Table 1. Comparison Of Number Of Solutions To The Assymetric FMTSP With The Equivalent One Salesman Problem

n \ m	1	2	3	4	5
3	2[1] (2)	1 (6)	-	-	-
4	6 (6)	6 (24)	1 (120)	-	-
5	24 (24)	36 (120)	12 (720)	1 (5040)	-
6	120 (120)	240 (720)	120 (5040)	20 (40320)	1 (362880)
7	720 (720)	1800 (5040)	1200 (40320)	300 (362880)	30 (3628800)
8	5040 (5040)	15120 (40320)	12600 (362880)	4200 (3628800)	630 (39916800)
9	40320 (40320)	141120 (362880)	141120 (3628800)	58800 (39916800)	11760 (479001600)

[1]Upper number is $\binom{n-1}{m}(n-2)!/(m-1)!$, lower number is $(n+m-2)!$

Table 2. Total Number of Distinct Tours For The Symmetric FMTSP

n \ m	1	2	3	4	5	6	7	8	9
2	1								
3	1	1							
4	3	3	1						
5	12	15	6	1					
6	60	90	45	10	1				
7	360	630	375	105	15	1			
8	2520	5040	3465	1155	210	21	1		
9	20160	45360	35280	13545	2940	378	28	1	
10	181440	453600	393120	170100	42525	6552	630	36	1

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper develops the number of feasible solutions (m-tours) for both the asymmetric and symmetric m-travelling salesmen problems. The proofs are constructive and it is shown how these results may be used to design an enumeration tree for incorporation within a general branch-and-bound algorithm. In addition, for the m-travelling salesmen problem defined over the Euclidean vector space in which the number of salesmen used is a decision variable, it is shown that there exists an optimal tour using only one salesman.

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